Bazinga! Maths 2019

Math and Physics Club, IIT Bombay

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1 Prelims

1. A student didn't notice a multiplication sign between two 3 digit numbers and wrote it as a single 6 digit number. The resultant number was 7 times more than what it should have been.
   Find the numbers.

   Answer: $143, 143$
   Solution:

   Let the two numbers be u and v.
   The problem is equivalent to solving $1000u + v = 7uv$.

   Hence, $u = \dfrac{v}{7v - 1000}$.

   As u is a three-digit number, we have that $100 \leq u \leq 999$.
   Solving the inequality gives $v = 143$ as the only integer solution.
   Substituting it back gives $u = 143$ as well.

2. An algebraic number is defined to be any real number that is a root of a non-zero polynomial (in one variable) with integer coefficients.
   Fill in the blank with the biggest subset A of $S = \{\sin 1°, \cos 1°\}$ such that all elements of A are algebraic. (A may be the empty set.)

   Answer: $\{\sin 1°, \cos 1°\}$ or S
   Solution:

   Claim: $\cos n\theta$ can be expressed as a polynomial in $\cos\theta$ with integer coefficients for $n \in \mathbb{N}$.
   Proof: It is clear that it is true for $n = 0, 1$.
   Assume that it is true for $n \leq k$.
   Let T_n be the polynomial with integer coefficients such that $T_n(\cos\theta) = \cos n\theta$.
   Observe:

$$\begin{aligned}
\cos(k+1)\theta = \cos[k\theta + \theta] &= \cos k\theta \cos\theta - \sin k\theta \sin\theta \\
&= \cos k\theta \cos\theta - \frac{1}{2}[\cos(k-1)\theta - \cos(k+1)\theta] \\
\implies \cos(k+1)\theta &= 2\cos k\theta \cos\theta - cos(k-1)\theta \\
&= 2T_k(\cos\theta)T_1(\cos\theta) - T_{k-1}(\cos\theta)
\end{aligned}$$

   By induction hypothesis, it is clear that the right hand side is a polynomial in $\cos\theta$ with integer coefficients.
   Thus, by principle of (strong) mathematical induction, we have proven the claim.

Now, $\cos 1°$ satisfies $T_{90}(x) = 0$.
Thus, $\cos 1°$ is algebraic. (It is obvious that T_{90} is not identically zero.)

Similarly, $\sin 1°$ is also a root of $T_{90}(x) = 0$. ($\sin 1° = \cos 89°$)

3. For $n \in \mathbb{N}$, let a_n denote the number of ordered pairs $(a, b) \in \mathbb{N}^2$ such that:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

Find the smallest n such that $a_n = 485$.
Submit your answer by writing n in its prime factorisation or by writing '∞', if no such n exists.

Answer: $2^{48} \cdot 3^2$
Solution:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

$$\implies ab = na + nb$$
$$\implies n^2 = (a - n)(b - n)$$

The number of solutions (a, b) to the above equation is the number of positive integer factors that n^2 has.
(The negative factors won't work as one of them would have magnitude greater than n, giving one of a or b as negative. In the case that both factors are $-n$, both a and b would be 0, again a contradiction.)

Let $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ be the prime factorisation of n.
Then, the number of factors of n^2 is $(2\alpha_1 + 1)(2\alpha_2 + 1) \cdots (2\alpha_n + 1)$.
485 can be written as a product of its factors in the following two ways:
$485 = 1 \times 485 = 5 \times 97$.
1×485 gives us that $\alpha_1 = 242$. The smallest such number would be $n = 2^{242}$.
5×97 gives us $\alpha_1 = 48$ and $\alpha_2 = 2$. The smallest such number would be $n = 2^{48} \cdot 3^2$.
It is easy to see that the smaller n is $\boxed{2^{48} \cdot 3^2}$.

4. If $a_1, a_2, \cdots, a_n$ are n distinct odd natural numbers, not divisible by an prime greater than 5, then find the minimum value of the constant L such that:

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq L \qquad \forall n \in \mathbb{N}$$

Answer: 15/8
Solution:
Given any a_i, its prime factorisation is $a_i = 3^{m_i} \cdot 5^{n_i}$ where m_i, n_i are non-negative integers.

To infinite sum S of reciprocals of all such possible a_is will converge to the following expression:

$$S = \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)\left(\frac{1}{1} + \frac{1}{5} + \frac{1}{5^2} + \cdots\right)$$

$$\implies S = \left(\frac{1}{1 - 1/3}\right)\left(\frac{1}{1 - 1/5}\right)$$

$$\implies S = \frac{15}{8}$$

As all the terms are positive, any finite sum will be less than $\frac{15}{8}$. Moreover, as the sum converges to $\frac{15}{8}$, it means that we can get arbitrarily close to $\frac{15}{8}$.

Therefore, the value of L is $\boxed{\frac{15}{8}}$.

5. Find the last 8 digits in the binary expansion of 27^{1986}.

Answer: 11011001
Solution:
The last 8 digits would be N converted to binary, where N is the remainder left when 27^{1986} is divided 2^8.
Note that $\phi(256) = 256\left(1 - \frac{1}{2}\right) = 128$, where ϕ represents the Euler totient function.
Thus, $27^{1986} \equiv 27^{66} \mod 256$.
Now,

$$\begin{aligned} 27^{66} &\equiv (32-5)^{66} \mod 256 \\ &\equiv 32^{66} - \binom{66}{1}32^{65}\cdot 5 - \cdots - \binom{66}{65}32\cdot 5^{65} + 5^{66} \mod 256 \\ &\equiv -\binom{66}{65}32\cdot 5^{65} + 5^{66} \mod 256 \qquad (32^3 \equiv 0 \mod 256) \end{aligned}$$

Note that $5^{64} \equiv 1 \mod 256$ since

$$5^{64} - 1 = (5-1)(5+1)(5^2+1)(5^4+1)(5^8+1)(5^{16}+1)(5^{32}+1)$$

The first term has a factor of 2^2 and the last 6 terms have a factor of 2. This gives us that $5^{64} \equiv 1 \mod 256$.
This gives us that $5^{65} \equiv 5 \mod 256$ and $5^{66} \equiv 25 \mod 256$. Thus,

$$\begin{aligned} -\binom{66}{65}32\cdot 5^{65} + 5^{66} &\equiv -(66)(32)(5) + 25 \mod 256 \\ &\equiv -(33)(64)(5) + 25 \mod 256 \\ &\equiv -(32)(64)(5) - (64)(5) + 25 \mod 256 \\ &\equiv 25 - 320 \mod 256 \\ &\equiv -295 \mod 256 \\ &\equiv 217 \mod 256 \end{aligned}$$

Thus, 217 is the required N.
This gives us the binary digits as $\boxed{11011001_2}$.

6. Let $S = \left\{ z \in \mathbb{C} \,\middle|\, \frac{\sqrt{2}}{2} \leq \mathrm{Re}(z) \leq \frac{\sqrt{3}}{2} \right\}$.
Find the smallest value of $p \in \mathbb{N}$ such that for all integers $n \geq p$, there exists $z \in S$ such that $z^n = 1$.

Answer: 16
Solution:

Let us find the largest integer q such that there exists no z in S with $z^q = 1$. As complex roots of unity satisfy $|z| = 1$, the problem is equivalent to there being no roots with argument between $\dfrac{\pi}{6}$ and $\dfrac{\pi}{4}$. (As these roots occur in conjugate pairs, it is sufficient to consider only the first quadrant.)
Note that the q^{th} roots have arguments $0, \frac{2\pi}{q}, \frac{4\pi}{q}, \cdots, \frac{2\pi k}{q}, \cdots, \frac{2\pi(q-1)}{q}$.
Note that if $q \geq 24$, then $\frac{2\pi}{q} \leq \frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$. Thus, there will exist some k such that $\dfrac{\pi}{6} \leq \frac{2\pi k}{q} \leq \dfrac{\pi}{4}$.
For q such that $16 \leq q < 24$, we have that $\frac{\pi}{6} < \frac{4\pi}{q} \leq \frac{\pi}{4}$. Therefore, there exists a root which is also in S.
Thus, we have shown that for all integers $q > 15$, we will always a $z \in S$ such that $z^q = 1$.
Now, observe that if $q = 15$, then $\frac{2\pi}{q} = \frac{2\pi}{15} < \frac{\pi}{6}$ and $\frac{4\pi}{q} = \frac{4\pi}{15} > \frac{\pi}{4}$. Thus, there is no $z \in S$ such that $z^{15} = 1$. This means that 15 is the desired value of q and the answer is $q + 1 = \boxed{16}$.

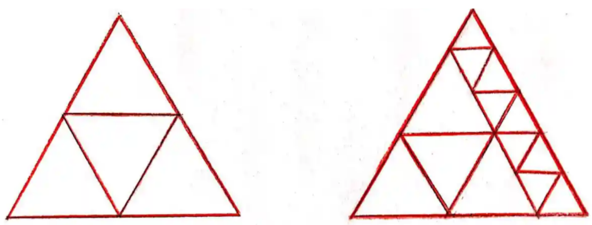
7. Let $\lfloor . \rfloor$ denote the greatest integer function.
Find the value of $\lfloor P \rfloor$ where P is given by:

$$P := \frac{\displaystyle\sum_{n=1}^{99} \sqrt{10 + \sqrt{n}}}{\displaystyle\sum_{n=1}^{99} \sqrt{10 - \sqrt{n}}}$$

Answer: 2
Solution: Define $S := \displaystyle\sum_{n=1}^{99} \sqrt{10 + \sqrt{n}}$ and $T := \displaystyle\sum_{n=1}^{99} \sqrt{10 - \sqrt{n}}$

$$
\begin{aligned}
\sqrt{2}S &= \sum_{n=1}^{99} \sqrt{20 + 2\sqrt{n}} \\
&= \sum_{n=1}^{99} \sqrt{20 + 2\sqrt{(10 + \sqrt{100 - n})(10 - \sqrt{100 - n})}} \\
&= \sum_{n=1}^{99} \sqrt{10 + \sqrt{100 - n} + 2\sqrt{(10 + \sqrt{100 - n})(10 - \sqrt{100 - n})} + 10 - \sqrt{100 - n}} \\
&= \sum_{n=1}^{99} \sqrt{\left(\sqrt{10 + \sqrt{100 - n}} + \sqrt{10 - \sqrt{100 - n}}\right)^2} \\
&= \sum_{n=1}^{99} \sqrt{10 + \sqrt{100 - n}} + \sqrt{10 - \sqrt{100 - n}} \\
&= \sum_{n=1}^{99} \sqrt{10 + \sqrt{n}} + \sqrt{10 - \sqrt{n}} & (n \mapsto 1 \\
&= S + T \\
\implies \sqrt{2}S &= S + T \\
\implies \frac{S}{T} &= \sqrt{2} + 1 = P \\
\therefore \lfloor P \rfloor &= \boxed{2}
\end{aligned}
$$

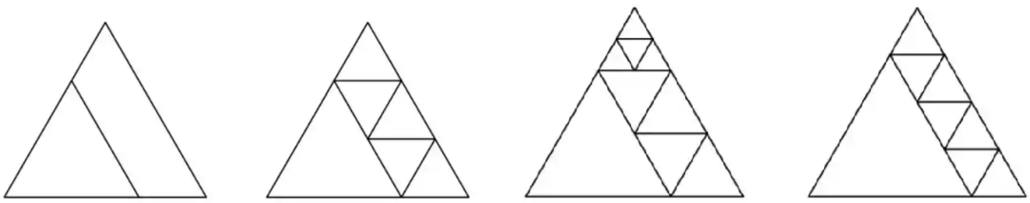
8. An equilateral triangle is divided into smaller equilateral triangles.
The figure shows that it is possible to divide it into 4 and 13 equilateral triangles. What are the integer values of n, where $n > 1$, for which it is possible to divide the triangle into n smaller equilateral triangles?



Answer: $\mathbb{N} \setminus \{1, 2, 3, 5\}$
Solution:

Let's call the equilateral triangle which is to be divided into smaller equilateral triangles the original triangle. Let's start with $n = 2$. Is there a way of dividing the original triangle into two smaller ones? We can't draw a line from a vertex, since this would produce angles less than $60°$. But if we draw a single line that draws an equilateral triangle, as in (a) below, we create another quadrilateral in the original triangle, not a triangle. So n cannot be 2.
If we were going to be able to divide the quadrilateral in (a) into equilateral triangles, we could only divide it into an odd number of equilateral triangles, as in (b), which divides

into 5 smaller equilateral triangles. Since there is no way to divide the remaining quadrilateral into an even number of triangles, we can't divide it into either 2 or 4 triangles, so we can also rule out $n = 3$ and $n = 5$.
The case for $n = 4$ is given in the question and so $n = 4$ is possible.
Diagram (b) shows the case for $n = 6$. Any of the smaller equilateral triangles in this diagram can be split into 4 equilateral triangles, which has the effect of adding 3 to the total number of triangles. (Diagram (c) is (b) with the top triangle divided into 4.) We can continue in this fashion in order to create cases when n $= 9, 12, 15, \ldots$ Therefore, it is possible to split the original triangle into n smaller equilateral triangles where n is a multiple of 3, (excluding $n = 3$).
Therefore, is is possible to divide the triangle into n smaller equilateral triangles for all positive integer values of n, excluding $2, 3$ and 5. The answer follows by noting that $n > 1$ was given.



(a) (b) (c) (d)

9. Find all the values of a in $\{2, 3, \ldots, 999\}$ for which $a^2 - a$ is divisible by 1000.

Answer: $376, 625$
Solution:

We have $1000 = 8 \times 125$ as product of prime powers. We have $1000 | a^2 - a$ if and only if $125 | a(a-1)$ and $8 | a(a-1)$. Because a and $a-1$ cannot share a factor, in turn this is equivalent to having both the conditions (1) $625 | a$ or $625 | a-1$ AND (2) $16 | a$ and $16 | a-1$. Now if the coprime integers 8 and 125 both divide the same natural number (in our case a or $a-1$), their product 1000 will also divide this number. In our case, this would force $a = 0, 1$, or ≥ 1000, all of which are not allowed. Thus, the given requirement on a is equivalent to having either (1) $8 | a$ and $125 | a-1$ OR (2) $8 | a-1$ and $125 | a$. Each case has a unique solution, respectively $a = \boxed{376}$ and $a = \boxed{625.}$
Using modular arithmetic to solve (1), we get: $a = 125k + 1 = 120k + (5k + 1)$. This gives us that $5k + 1 \equiv 0 \mod 8$ or $3k \equiv 1 \mod 8$, forcing $k = 3$, given the constraint. Similarly, (2) can be solved.

10. All the terms in the sequence (a_n) are positive real numbers and the sequence (a_n) satisfies the equation below for all positive integers n :

$$\sum_{k=1}^{n} a_k = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right).$$

Find the value of

$$\sum_{k=1}^{100} a_k.$$

Answer: 10
Solution:
Let $s_n = \sum_{k=1}^{n} a_k$. Then we have

$$\begin{aligned} s_{n+1} &= \frac{1}{2}\left(a_{n+1} + \frac{1}{a_{n+1}}\right) \\ 2(s_n + a_{n+1}) &= a_{n+1} + \frac{1}{a_{n+1}} && [s_{n+1} = s_n + a_{n+1}] \\ a_{n+1}^2 + 2s_n a_{n+1} - 1 &= 0 \\ a_{n+1} &= \sqrt{s_n^2 + 1} - s_n \\ s_{n+1} &= \sqrt{s_n^2 + 1} \\ s_{n+1}^2 &= s_n^2 + 1 \end{aligned}$$

Since $s_1 = a_1$, we have $a_1 = \frac{1}{2}\left(a_1 + \frac{1}{a_1}\right) \implies a_1 = 1 = s_1$. Then we have:

$$\begin{aligned} s_2^2 &= s_1^1 + 1 = 2 \\ s_3^2 &= s_2^1 + 1 = 3 \\ &\vdots \\ s_{100}^2 &= 100 \end{aligned}$$

Thus, $\sum_{k=1}^{100} a_k = s_{100} = \boxed{10}$.

2 Rapid Fire

Q 2.1. Find the sum of all natural numbers a such that $a^2 - 16a + 67$ is a perfect square.

Answer: 16

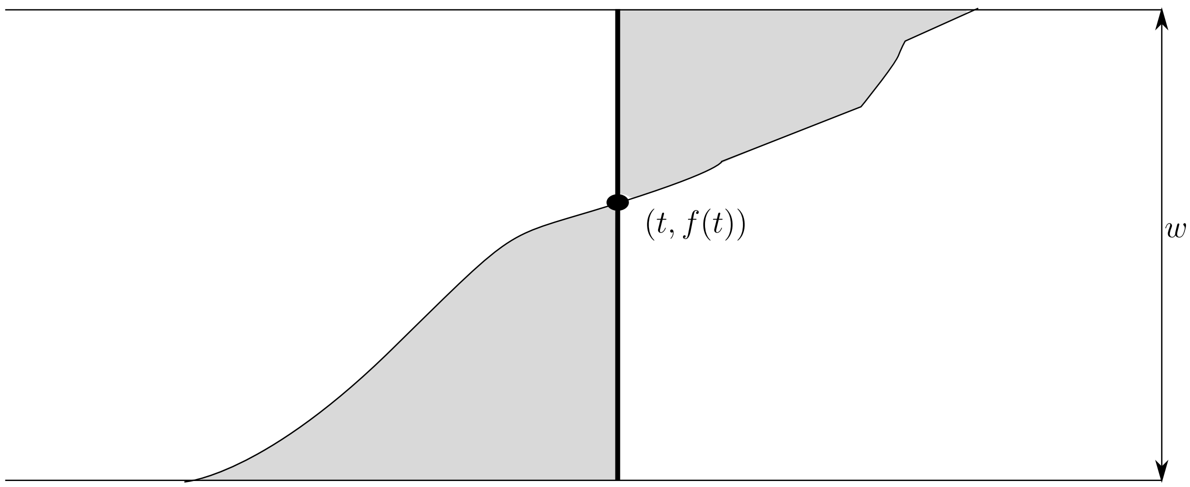
By hypothesis,

$$\begin{aligned} a^2 - 16a + 67 &= p^2 && \text{For some } p \in \mathbb{Z} \\ \implies (a-8)^2 - p^2 &= -3 \\ \implies (a-8-p)(a-8+p) &= -3 \end{aligned}$$

The only ways to factorise -3 into two factors along with the corresponding pairs (a, p) are:
1. $1 \times -3 \longrightarrow (7, -2)$
2. $-1 \times 3 \longrightarrow (9, 2)$
3. $-3 \times 1 \longrightarrow (7, 2)$
4. $3 \times -1 \longrightarrow (9, -2)$

Thus, the desired sum is $7 + 9 = \boxed{16}$.

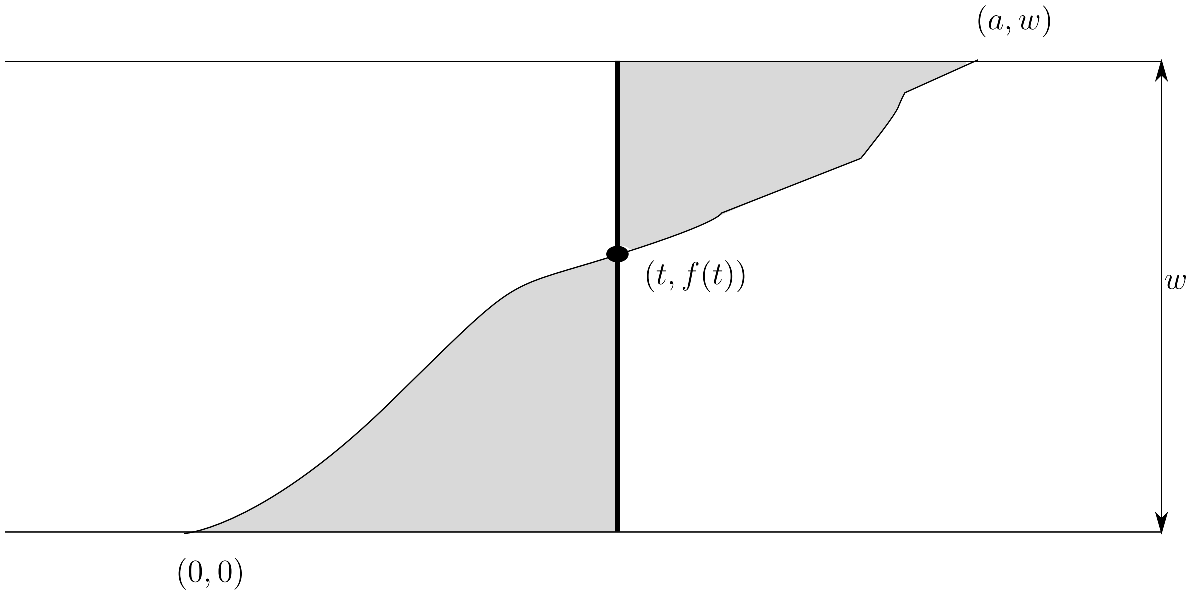
Q 2.2. The graph of a monotonically increasing continuous function is cut off with two horizontal lines. Find a point on the graph between intersections such that the sum of the two areas bounded by the lines, the graph and the vertical line through the point is minimum. See figure.



Minimise the area

Answer: The point on the graph that lies on the middle line between and parallel to the two lines.

Let us first label the start and end points. We may assume the start point to be $(0, 0)$ and the end-point to be (a, w). This is consistent with the fact that the distance between the lines is w.



Let $(t, f(t))$ be the point which minimises the area. We may write area as a function of t,

$$A(t) = \int_0^t f(x)dx + \int_t^a (w - f(x))dx$$

As we wish to minimise this area, we'll set $\frac{d}{dt}A(t) = 0$

$$A'(t) = f(t) - w + f(t) \qquad \text{(Using Fundamental Theorem of Calculus)}$$
$$A'(t) = 2f(t) - w$$
$$\therefore A'(t) = 0 \implies f(t) = \frac{w}{2}$$

Thus, the required point is the point on the function that lies on the middle line between and parallel to the two lines.
To see that this is in fact a minimum, note that f is an increasing function. So, the derivative $A'(t)$ goes from negative to positive giving us the required minimum.
Note that we cannot differentiate $A'(t)$ and use the second derivative test as f may not be differentiable. Also, we could use the Fundamental Theorem of Calculus as f was continuous.

Q 2.3. Find the remainder left when $(2500)(98!)^2$ is divided by 101.

Answer: 19
Since 101 is prime, $100! \equiv -1 \mod 101$, by Wilson's Theorem.

$$\begin{aligned} 100! &\equiv -1 \mod 101 \\ \implies 100! &\equiv 100 \mod 101 \\ \implies 99! &\equiv 1 \mod 101 \\ \implies 98! &\equiv 99^{-1} \mod 101 \end{aligned}$$

To calculate the modular inverse of 99, observe the following:

$$\begin{aligned} 99 &\equiv -2 \mod 101 \\ \implies 99 \cdot 50 &\equiv -2 \cdot 50 \mod 101 \\ \implies 99 \cdot 50 &\equiv -100 \mod 101 \\ \implies 99 \cdot 50 &\equiv 1 \mod 101 \\ \implies 99^{-1} &\equiv 50 \mod 101 \end{aligned}$$

This gives us that $98! \equiv 50 \mod 101$.
Therefore, $(98!)^2 \equiv 2500 \equiv -25 \mod 101$.
This gives us $(2500)(98!)^2 \equiv (2500)(-25) \equiv 625 \equiv \boxed{19} \mod 101$.

Q 2.4. Find the following limit:

$$\lim_{x \to 0} \frac{\sin \tan \arcsin x - \tan \sin \arctan x}{\tan \arcsin \arctan x - \sin \arctan \arcsin x}$$

Answer: 1
We have to calculate a limit of the form

$$\lim_{x \to 0} \frac{f(x) - g(x)}{g^{-1}(x) - f^{-1}(x)}.$$

Details are left to the reader. (Think about the graph and the behaviour near 0.)

Q 2.5. Find the minimum value of $|\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$ for real x.

Answer: $2\sqrt{2} - 1$
Put $x = y - \frac{3\pi}{4}$. Then, $\sin x = -(\cos y + \sin y)/\sqrt{2}$, $\cos x = -(\cos y - \sin y)/\sqrt{2}$, so $\sin x + \cos x = -\sqrt{2}\cos y$.
Similarly, $\tan x + \cot x = (\sin x \cos x)^{-1} = 2(\cos^2 y - \sin^2 y)^{-1} = 2(2\cos^2 y - 1)^{-1}$.
Also, $\sec x + \operatorname{cosec} x = -2\sqrt{2}\cos y(2\cos^2 y - 1)^{-1}$.
Substituting $\sqrt{2}\cos y = c$, we get:

$$\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x = -c - \frac{2}{c+1} = -(c+1) - \frac{2}{c+1} + 1$$

If $c + 1$ is positive, we have:

$$c + 1 + \frac{2}{c+1} \geq 2\sqrt{2} \qquad (AM \geq GM) \qquad (1)$$

$$-(c+1) - \frac{2}{c+1} \leq -2\sqrt{2} \qquad (2)$$

$$-c - \frac{2}{c+1} \leq 1 - 2\sqrt{2} < 0 \qquad (3)$$

$$\therefore \left|-c - \frac{2}{c+1}\right|_{\min} = 2\sqrt{2} - 1 \qquad \text{(After verifying that equality is in fact possible.)} \qquad (4)$$

If $c + 1$ is negative, we have:

$$-(c+1) - \frac{2}{c+1} \geq 2\sqrt{2} \qquad (5)$$

$$-c - \frac{2}{c+1} \geq 1 + 2\sqrt{2} > 0 \qquad (6)$$

$$\therefore \left|-c - \frac{2}{c+1}\right|_{\min} = 2\sqrt{2} + 1 \qquad (7)$$

As $2\sqrt{2} - 1 < 2\sqrt{2} + 1$, the final answer is $\boxed{2\sqrt{2} - 1}$.

Q 2.6. Let's start with an ordered pair of integers, (a, b). We start a process, where we keep applying the following transformation T :

$$T(a,b) = \begin{cases} (2a, b-a) & \text{if } a < b \\ (a-b, 2b) & \text{otherwise} \end{cases}$$

We say that the process terminates for a given pair (a, b) if there exists an $n \in \mathbb{N}$ such that $T^n(a, b) = T^{n+1}(a, b)$,
where T^n denotes the composition of T with itself n times.
For which of the given pairs does the process terminate?

1. (24, 101)
2. (97, 31)
3. (34, 56)
4. (19, 79)

Answer: $(97, 31)$
It is not tough to see that $T(a, b) = (a, b)$ iff $a = 0$ or $b = 0$.
Also, observe that sum of the elements of the ordered pair does not change under the transformation. We shall denote this sum by s.
Therefore, the process terminates iff $T^n(a, b) = (0, s)$ or $(s, 0)$.

Now, let us observe the first element of the ordered pair.
If $a < n/2$, then $a \mapsto 2a$, else $a \mapsto 2a - s$.
Thus, if we look at the sequence of remainders left when the first element is divided by s, we get the sequence:

$$a, 2a \bmod s, 2^2 a \bmod s, 2^3 a \bmod s, \cdots$$

As the process terminates iff the first element is 0 or n, the remainder must be 0.
Thus, there must exist $k \in \mathbb{N}$ such that $2^k a \equiv 0 \mod n$.
It is easy to see that $\boxed{(97, 31)}$ is the only such pair with $k = 7$.

Q 2.7. $\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} \times 3 = \sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}$ where a, b, c are positive integers. What is the value of $a+b+c$?

Answer: 47

Let x, y, v be positive reals such that $x^3 = 5, y^3 = 4, v^3 = 2$. So $y = v^2, y^2 = 2v, vy = 2$.

$$(x-y)(x+y)^2 = x^3 - y^3 + x^2y - xy^2 = 1 + (xv)^2 - 2xv = (xv-1)^2$$

Hence,

$$\begin{aligned} 9\sqrt{x-y} &= \sqrt{(x-y)}(x^3+y^3) = \sqrt{(x-y)(x+y)^2}(x^2-xy+y^2) \\ &= (xv-1)(x^2-xy+y^2) = x^3v - x^vy + xvy^2 - x^2 + xy - y^2 \\ &= 3v + 3xy - 3x^2 = 3(v+xy-x^2) \\ \implies 3\sqrt{x-y} &= v + xy - x^2 \end{aligned}$$

That is,

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} \times 3 = \sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}$$

This gives us that $a+b+c = 2+20+25 = \boxed{47}$

Q 2.8. Let S be a finite non-empty set of real positive numbers.
If S contains at least n elements, it is guaranteed that there are elements $x, y \in S$ such that

$$0 < \frac{y-x}{1+xy} < \sqrt{2} - 1$$

What is the smallest such n?

Answer: 5
We have it that $x < y$. Let $x = \tan\alpha$ and $y = \tan\beta$ where $0 < \alpha < \beta < \frac{\pi}{2}$.
Note that $\tan : \left(0, \dfrac{\pi}{2}\right) \longrightarrow \mathbb{R}^+$ is an order preserving bijection.
With this substitution, we have

$$\frac{y-x}{1+xy} = \tan(\beta - \alpha).$$

We need $\tan(\beta - \alpha) < \tan\left(\dfrac{\pi}{8}\right)$.

This is equivalent to $\beta - \alpha < \dfrac{\pi}{8}$.

Now, note that that interval $\left(0, \dfrac{\pi}{2}\right)$ can be written as a disjoint union of intervals in the following manner:

$$\left(0, \frac{\pi}{2}\right) = \left(0, \frac{\pi}{8}\right] \cup \left(\frac{\pi}{8}, \frac{\pi}{4}\right] \cup \left(\frac{\pi}{4}, \frac{3\pi}{8}\right] \cup \left(\frac{3\pi}{8}, \frac{\pi}{2}\right)$$

If $n \geq 5$, there would have to be two distinct elements which are in the same interval. (By PHP). This would ensure that their difference is less than $\pi/8$ and thus, we'd have x and y satisfying the given inequality.
Now, we have to show that if $n = 4$, there does exist a set S such that no two distinct elements of the set satisfy the inequality. This can be easily demonstrated by taking the set: $S = \left\{\tan\left(\dfrac{\pi}{16}\right), \tan\left(\dfrac{3\pi}{16}\right), \tan\left(\dfrac{5\pi}{16}\right), \tan\left(\dfrac{7\pi}{16}\right)\right\}$.

Thus, the smallest such n is $\boxed{5}$.

Q 2.9. Starting with the vertices $P_1 = (0,1),\ P_2 = (1,1)\ P_3 = (1,0),\ P_4 = (0,0)$ of a square, we construct further points as follows:

$$P_n \text{ is the midpoint of the line segment } \overline{P_{n-4}P_{n-3}} \text{ for } n \geq 5.$$

The spiral approaches a point $P = \lim\limits_{n\to\infty} P_n$. Find P.

Answer: $\left(\frac{4}{7}, \frac{3}{7}\right)$

Let $P_n = (x_n, y_n)$.
Note that $P = \lim\limits_{n\to\infty} P_n = \left(\lim\limits_{n\to\infty} x_n,\ \lim\limits_{n\to\infty} y_n\right)$.

By definition, we have that $x_n = \frac{1}{2}(x_{n-4} + x_{n-3})$ and $y_n = \frac{1}{2}(y_{n-4} + y_{n-3})$ for $n \geq 5$.

We can prove via induction that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$.

The case $n = 1$ is immediate as $\frac{1}{2} \cdot 0 + 1 + 1 + 0 = 2$.

Assume that the result holds for $n = k - 1 \geq 1$, then:

$$\begin{aligned}
\frac{1}{2}x_k + x_{k+1} + x_{k+2} + x_{k+3} &= \frac{1}{2}x_k + x_{k+1} + x_{k+2} + \frac{1}{2}(x_{k-1} + x_k) && (\because k + 3 \geq 5) \\
&= \frac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} \\
&= 2 && \text{(by induction hypothesis)}
\end{aligned}$$

Assuming convergence, we may now write:

$$\lim_{n\to\infty}\left(\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}\right) = \frac{7}{2} \cdot \lim_{n\to\infty} x_n = 2$$
$$\implies \lim_{n\to\infty} x_n = \frac{4}{7}.$$

A similar argument works for y_n with the invariant $\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2}$.
This gives us $\lim\limits_{n\to\infty} y_n = \frac{3}{7}$.

Thus, $P = \boxed{\left(\frac{4}{7}, \frac{3}{7}\right)}$.

Note that we have simply assumed convergence. It is not too tough to show that x_n and y_n actually do converge.

Q 2.10. Arrange the integers $1, 2, 3, \cdots, 10$ in some order, and get the sequence $a_1, a_2, a_3, \cdots, a_{10}$. The sequence satisfies that the unit digit of $a_n + n$ are all different for $n = 1, 2, 3, \cdots, 10$. How many such arrangements are possible?

Answer: 0
We are adding all the integers $1, 2, 3, \cdots 10$ at least once in some order, and adding all the ns of $a_1, a_2, a_3, \cdots a_{10}$ in dome order, for a total of $1+2+3+\cdots+10+1+2+3+\cdots+10 = 90$, so the sum of all the unit digits of $a_n + n$ must end in 0.
However, since all 10 unit digits of $a_n + n$ must be different, all 10 unit digits $0, 1, 2, \cdots, 9$ are represented, but, $0+1+2+\cdots+9 = 45$, which ends in 5, not 0.
Therefore, $\boxed{0}$ arrangements are possible.

Q 2.11. In how many ways can we transform $f(x) = x^2 + 4x + 3$ into $g(x) = x^2 + 10x + 9$ by a sequence of transformations of the form

$$f(x) \mapsto x^2 f\left(\frac{1}{x} + 1\right)$$

or

$$f(x) \mapsto (x-1)^2 f\left(\frac{1}{x-1}\right)?$$

(You may answer with infinite as well.)

Answer: 0
Observe that both the transformations keep the discriminant invariant. As f and g have distinct determinants, it is not possible. Thus, the answer is $\boxed{0}$.

Q 2.12. You are given a rope of length 1. You pick a real number x randomly from $(0,1)$ with uniform distribution. You then cut as many segments of length x as possible. In other words, you cut the length nx where n is the largest integer such that $nx \leq 1$.
What is the expected length of the rope remaining?

Answer: $1 - \dfrac{\pi^2}{12}$

Let $L : (0,1) \longrightarrow \mathbb{R}$ be a function such that $L(x)$ denotes the length of the rope left when x is chosen. It is easy to see that:

$$L(x) = 1 - \left\lfloor \frac{1}{x} \right\rfloor x$$

The expected length will be given by:

$$\frac{\displaystyle\int_0^1 L(x)dx}{\displaystyle\int_0^1 dx}.$$

The denominator is simply 1, the numerator can be evaluated as follows:

$$\begin{aligned}
\int_0^1 L(x)dx &= \int_{1/2}^1 (1-x)\mathrm{d}x + \int_{1/3}^{1/2} (1-2x)\mathrm{d}x + \int_{1/4}^{1/3} (1-3x)\mathrm{d}x + \cdots \\
&= \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} (1-nx)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} - \frac{n}{2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{2n} + \frac{n}{2(n+1)^2} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{n+1} + \frac{n+1-1}{2(n+1)^2} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+1)} - \frac{1}{2(n+1)^2} \right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{1}{(n+1)^2} \right) \\
&= \frac{1}{2}(1) - \frac{1}{2}\left(\frac{\pi^2}{6} - 1 \right) \\
&= \boxed{1 - \frac{\pi^2}{12}}
\end{aligned}$$

Q 2.13. For an arbitrary n, let $g(n)$ be the GCD of $2n+1$ and $2n^2+7n+17$. What is the largest positive integer that can be obtained as the value of $g(n)$? If $g(n)$ can be arbitrarily large, state so explicitly.

Answer: 7
Long division gives $2n^2+7n+17 = (2n+1)(n+3)+14$.
By Euclidean algorithm, $\text{GCD}(2n^2+7n+17, 2n+1) = \text{GCD}(2n+1, 14)$.
Thus, $g(n)$ divides 14. But since $g(n)$ dives $2n+1$, which is odd, $g(n)$ divides 7. Therefore, $g(n) \leq 7$.
The fact that 7 can actually be obtained follows from $n = 3 \implies 2n+1 = 7$.

Q 2.14. Let $x \in \mathbb{C}$ be such that

$$x + x^{-1} = \frac{\sqrt{5}+1}{2}.$$

What is the value of $x^{2019} + x^{-2019}$?

Answer: $\dfrac{\sqrt{5}+1}{2}$
Given, $x + x^{-1} = 2\cos\left(\frac{\pi}{5}\right)$.
By inspection, the solutions to the above equation are $\exp\left(\pm i\frac{\pi}{5}\right)$. As the solutions are reciprocals, we can take any one solution and evaluate $x^{2019} + x^{-2019}$.
Taking $x = e^{i\pi/5}$, we have $x^{2019} = e^{2019i\pi/5}$.
Thus, $x^{2019} + x^{-2019} = 2\cos\left(\frac{2019\pi}{5}\right) = 2\cos\left(\frac{\pi}{5}\right) = \boxed{\frac{\sqrt{5}+1}{2}}$.

Q 2.15. 100 numbers $1, 1/2, 1/3, \cdots 1/100$ are written on the blackboard. One may delete two arbitrary numbers a and b among them and replace them by the number $a+b+ab$. After 99 such operations, only one number is left. What are all the possible values of the final number?

Answer: 100
The idea is to look for a function of the numbers on the blackboard which remains *invariant* when two numbers a, b are replaced by a single number $a+b+ab$. Now

$$(1+a)(1+b) = (1+a+b+ab),$$

so a function which remains invariant is the product

$$P = (1+a_1)(1+a_2)\cdots(1+a_n),$$

where $a_1, a_2, \ldots, a_n$ are the numbers currently on the blackboard. Initially,

$$\begin{aligned} P &= (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{1000}\right) \\ &= (2)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{100}{99}\right)\left(\frac{101}{100}\right) \\ &= 101. \end{aligned}$$

Finally,

$$P = 1 + A$$

where A is the final number on the board. Thus, $A = \boxed{100}$ is only possibility.

Q 2.16. A function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfies $f(m+n) = mn + f(m) + f(n) + 1$.

Find the value of $\sum_{n=-24}^{24} f(n)$.

Answer: 4851
$P(m,n) : f(m+n) = mn + f(m) + f(n) + 1$
$P(0,0) \implies f(0) = -1$
$P(-n,n) \implies f(-n) + f(n) = n^2 - 2$
Now, we have

$$\begin{aligned}\sum_{n=-24}^{24} f(n) &= f(0) + \sum_{n=1}^{24}(f(-n) + f(n)) \\ &= f(0) + \sum_{n=1}^{24}(n^2 - 2) \\ &= f(0) + \frac{(24)(25)(49)}{6} - 2(24) \\ &= -1 + 4900 - 48 \\ &= \boxed{4851}\end{aligned}$$

Q 2.17. The sequence (a_n) is defined as follows: $a_1 = 1,\ a_2 = 2$ and

$$a_{n+2} = \frac{2}{a_{n+1}} + a_n \text{ for } n \geq 1.$$

What is $a_{100} \cdot a_{101}$?

Answer: 200
Re-arranging gives us: $a_{n+2}a_{n+1} - a_{n+1}a_n = 2$.

$$\begin{aligned}\sum_{n=1}^{99}(a_{n+2}a_{n+1} - a_{n+1}a_n) &= \sum_{n=1}^{99} 2 \\ \implies a_{101}a_{100} - a_2a_1 &= 198 \\ \implies a_{101}a_{100} &= \boxed{200}\end{aligned}$$

Q 2.18. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and $T_n = \frac{1}{(n+1)H_nH_{n+1}}$.

Evaluate $\sum_{n=1}^{\infty} T_n$.

Answer: 100
Observe that $H_{n+1} = H_n + \frac{1}{n+1}$, or $\frac{1}{n+1} = H_{n+1} - H_n$.

$$T_n = \frac{1}{n+1}\frac{1}{H_nH_{n+1}} = \frac{H_{n+1} - H_n}{H_nH_{n+1}} = \frac{1}{H_n} - \frac{1}{H_{n+1}}.$$

Thus, we have $\sum_{k=1}^{n} T_n = \frac{1}{H_1} - \frac{1}{H_{n+1}}$.

As the harmonic series diverges, $\lim_{n\to\infty} \frac{1}{H_n} = 0$ giving us $\sum_{n=1}^{\infty} T_n = \boxed{1}$.

Q 2.19. Evaluate the following expression and give your answer in the simplest form:

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{\cdots+\cfrac{1}{2019}}}}} + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{\cdots+\cfrac{1}{2019}}}}}}.$$

Answer: 1

Let $x = \cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{\cdots+\cfrac{1}{2019}}}}$.

Then, the expression reduces to $\frac{1}{2+x} + \frac{1}{1+\frac{1}{1+x}} = \frac{1}{2+x} + \frac{1+x}{2+x} = \boxed{1}$.

Q 2.20. Six numbers are placed on a circle. For every number A on the circle, we have: $A = |B - C|$, where B and C follow A clockwise. The total sum of the numbers equal 1. State the numbers in cyclic order, starting with the smallest.

Answer: 0, 1/4, 1/4, 0, 1/4, 1/4

Let the numbers be A, B, C, D, E and F in clockwise order. They are all non-negative. We may assume that A is no smaller than any of the other five. We have either $A = B - C$ or $A = C - B$.

Suppose $A = B - C$. Then, we must have $B = A$ and $C = 0$.

$\implies F = |A - B| = 0 \implies E = |A - F| = A \implies D = |E - F| = A$.

If $A = C - B$, then $C = A$ and $B = 0$.

$\implies F = |A - B| = A \implies E = |A - F| = 0 \implies D = |E - F| = A$.

In either case, two of the numbers are 0 and are diametrically opposite each other. The other four are equal to each other. Since their sum is 1, each is 1/4.

Thus, the required sequence is - 0, 1/4, 1/4, 0, 1/4, 1/4.

Q 2.21. Find the smallest positive real number c such that the following inequality holds for all non-negative reals x and y :

$$\sqrt{xy} + c|x - y| \geq \frac{x+y}{2}$$

Answer: $\frac{1}{2}$

If $x = y$, it is clear that any value of c works. Assume $x > y$. Then we have:

$\sqrt{xy} + c(x - y) \geq \frac{x+y}{2}$

$\implies 2c(x - y) \geq \left(\sqrt{x} - \sqrt{y}\right)^2$

$\implies 2c \geq \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} = 1 - \frac{2\sqrt{y}}{\sqrt{x} + \sqrt{y}}$

Thus, any $2c \geq 1$ will satisfy the inequality. If $y = 0$, then $c = 1/2$ is actually required. Thus, $c = 1/2$ is the minimum positive real.

Q 2.22. Solve the following inequality for positive x :

$$x(8\sqrt{1-x} + \sqrt{1+x}) \leq 11\sqrt{1+x} - 16\sqrt{1-x}.$$

Answer: $\left[\frac{3}{5}, 1\right]$

First, observe that for $x > 1$, some of the terms become undefined. Thus, we must have that $x \leq 1$.

Define y as follows:

$$y = \sqrt{\frac{1-x}{1+x}}.$$

Observe that:

$$x = \frac{1-y^2}{1+y^2}.$$

Given the above preliminaries, our inequality transforms as follows:

$$\begin{aligned} x(8\sqrt{1-x} + \sqrt{1+x}) &\leq 11\sqrt{1+x} - 16\sqrt{1-x} \\ x\left(8\sqrt{\frac{1-x}{1+x}} + 1\right) &\leq 11 - 16\sqrt{\frac{1-x}{1+x}} \\ \frac{1-y^2}{1+y^2}(8y+1) &\leq 11 - 16y \\ (1-y^2)(8y+1) &\leq (11-6y)(1+y^2) \qquad (\because 1 + y^2 > 0) \\ -8y^3 - y^2 + 8y + 1 &\leq -16y^3 + 11y^2 - 16y + 11 \\ -8y^3 + 12y^2 - 24y + 10 &\geq 0 \\ (2y-1)\left(4y^2 - 4y + 10\right) &\leq 0 \end{aligned}$$

Now, $4y^2 - 4y + 10$ is always positive. Thus, the inequality reduces simply to: $2y - 1 \leq 0$.

Observe the fact that y is monotonically decreasing in x and $\frac{1-(1/2)^2}{1+(1/2)^2} = \frac{3}{5}$.

Our final answer is hence, $x \in \left[\frac{3}{5}, 1\right]$.

Q 2.23. Let p be an odd prime. Let x and y with $x < y$ be positive integers that satisfy

$$2xy = (x + p)(y + p).$$

What is the sum of all possible values of x?

Answer: $4p + 3$
$2xy = (x + p)(y + p) \iff 2xy = xy + px + py + p^2 \iff xy - px - py + p^2 = 2p^2$
$\iff (x - p)(y - p) = 2p^2$.
As $p > 2$, $2p^2$ has 6 distinct positive factors which are $1, 2, p, 2p, p^2, 2p^2$.
As $0 < x < y$, the values that $x-p$ can take are $1, 2, p$. This gives us the values of x as $p+1, p+2$ and $2p$. Thus, the sum is $\boxed{4p + 3}$.

Q 2.24. For any positive integer n, let $s(n)$ denote the number of ways that n can be written as a sum of 1s and 2s, where the order matters.
As an example, $s(3) = 3$ as $3 = 1 + 1 + 1 = 1 + 2 = 2 + 1$.
Evaluate

$$\lim_{n\to\infty} \frac{s(n-1)}{s(n)}.$$

Answer: $\dfrac{\sqrt{5}-1}{2}$ It is clear that $s(1) = 1$ and $s(2) = 2$.
Let us try to evaluate $s(n)$ for $n \geq 3$.
Given any sum, the last number in the sum is either 1 or 2. How many sums are there are end with 1? This is simple to calculate as the remaining numbers must add up to $n - 1$ and this can be done in $s(n - 1)$ ways. Similarly, there are $s(n - 2)$ sums that end with a 2.
As these two cases are exclusive and exhaustive, we have it that $s(n) = s(n - 1) + s(n - 2)$.
This is nothing but the Fibonacci sequence, just shifted.
Thus, we have it that

$$\lim_{n\to\infty} \frac{s(n-1)}{s(n)} = \left(\frac{\sqrt{5}+1}{2}\right)^{-1} = \frac{\sqrt{5}-1}{2}.$$

3 Brief Thought

Q 3.1. Prove that the probability that an integer is prime is 0. In other words, prove that if $\pi(N)$ denotes the number of primes $\leq N$, then

$$\lim_{N\to\infty} \frac{\pi(N)}{N} = 0.$$

Denote the first m primes by $p_1, p_2, \cdots, p_m$. We will choose the value of m later.
The number of positive integers $\leq N$ which are not divisible by any of the primes $p_1, p_2, \cdots, p_m$ is:

$$\begin{aligned} N - \left\lfloor \frac{N}{p_1} \right\rfloor - \left\lfloor \frac{N}{p_2} \right\rfloor - \cdots - \left\lfloor \frac{N}{p_m} \right\rfloor + \left\lfloor \frac{N}{p_1 p_2} \right\rfloor + \cdots + \left\lfloor \frac{N}{p_{m-1} p_m} \right\rfloor \\ - \left\lfloor \frac{N}{p_1 p_2 p_3} \right\rfloor - \cdots + (-1)^m \left\lfloor \frac{N}{p_1 p_2 \cdots p_m} \right\rfloor \end{aligned}$$

The primes p such that $p_m < p \leq N$ are not divisible by $p_1, \cdots p_m$ and are therefore counted in the above expression.
Since there are $\pi(N) - m$ such primes p, we have

$$\pi(N) \leq m + N - \left\lfloor \frac{N}{p_1} \right\rfloor - \cdots - \left\lfloor \frac{N}{p_m} \right\rfloor + \cdots + (-1)^m \left\lfloor \frac{N}{p_1 \cdots p_m} \right\rfloor \tag{1}$$

On the right side of (1), there are $\binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m} = 2^m - 1$ brackets, of which $\binom{m}{1} + \binom{m}{3} + \binom{m}{5} + \cdots = 2^{m-1}$ are preceded by a minus sign. Now suppose we remove all the brackets in (1). Since $\lfloor a \rfloor \leq a$ for any number a, the positive terms will be increased. And since $\lfloor a \rfloor > a - 1$, each negative term will be decreased by at most 1.

Therefore

$$\begin{aligned} \pi(N) &\leq m + 2^{m-1} + N - \frac{N}{p_1} - \cdots - \frac{N}{p_m} + \frac{N}{p_1 p_2} + \cdots + (-1)^m \frac{N}{p_1 \cdots p_m} \\ &= m + 2^{m-1} + N \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right). \end{aligned}$$

Since $m \leq 2^{m-1}$, we have

$$\pi(N) \leq 2^m + N \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right)$$

and therefore

$$\frac{\pi(N)}{N} \leq \frac{2^m}{N} + \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right)$$

Now, we choose m in such a way that as N tends to infinity, m also tends to infinity but $2^m/N \to 0$. For example, such a choice of m would be $\left\lfloor \log_2 \sqrt{N} \right\rfloor$.
To complete the proof, we only need to show that

$$\lim_{m\to\infty} \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right) = 0$$

By the formula for the sum of a geometric progression, we have

$$\frac{1}{1-\frac{1}{p}} > \frac{1-\frac{1}{p^{k+1}}}{1-\frac{1}{p}} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \frac{1}{p^k}$$

where k is any positive integer. Therefore

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_m}\right) > \left(1+\frac{1}{p_1}+\frac{1}{p_1^2}+\cdots\frac{1}{p_1^k}\right)$$
$$\times\left(1+\frac{1}{p_2}+\frac{1}{p_2^2}+\cdots\frac{1}{p_2^k}\right)\times\cdots\times\left(1+\frac{1}{p_m}+\frac{1}{p_m^2}+\cdots\frac{1}{p_m^k}\right)$$

Expanding this last product, we obtain a sum of fractions all having numerator 1 but various integers for their denominators. If $k = m$, every integer $\leq m$ will appear as one of these denominators, because every integer $\leq m$ can be factored into a product of powers of $p_1, p_2, \cdots m$. Therefore

$$\frac{1}{\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_m}\right)} > 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{m}$$

and so

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_m}\right) < \frac{1}{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{m}} \tag{2}$$

As $m \to \infty$, the right hand side of (2) tends to 0.
It is clear that the left hand side is always positive. Therefore, by Sandwich Theorem, we have it that

$$\lim_{m\to\infty}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_m}\right) = 0,$$

completing the proof.

Q 3.2. The circumference of a circle is divided into p equal parts by the points $A_1, A_2, \cdots A_p$, where p is an odd prime number. How many different self-intersecting p-gons are there with these points as vertices if two p-gons are considered different only when neither of them can be obtained from the other by rotating the circle? (A self-intersecting polygon is a polygon some of whose sides intersect at other points besides the vertices).

Answer: $N = \frac{1}{2}\left\{\frac{(p-1)!+1}{p} + p - 4\right\}$

Let the points be $A_0, A_1, A_2, \cdots, A_{p-1}$ going around the circle in a the counterclockwise direction. Put $A_p = A_0, A_{p+1} = A_1, \cdots, A_{2p} = A_0, A_{2p+1} = A_1$, etc. Let us first compute how many self-intersecting polygons there are with vertices at $A_0, A_1, \cdots A_{p-1}$, counting two polygons as different if they are different in shape or location. To obtain a polygon, we join A_0 to any other point A_{i_1} other than A_0, then join A_{i_1} to any point A_{i_2} other than A_0 and A_{i_1}, and continue in this way until all the points are exhausted; then we join the last point $A_{i_{p-1}}$ to A_0. The point A_{i_1} can be chosen in $(p-1)$ ways; once it is chosen, A_{i_2} can be chosen in $(p-2)$ ways and so on.
Therefore, the total number of ways of choosing the sequence $A_{i_1}, A_{i_2}, \cdots A_{i_{p-1}}$ is $(p-1)!$.
Note, however, each polygon is obtained exactly twice in this way as $A_0 A_{i_1} A_{i_2} \cdots A_{i_{p-1}} A_0$ is the same as $A_0 A_{i_{p-1}} A_{i_{i-1}} \cdots A_{i_1} A_0$ but it differs from all other $p-$gons.
Thus, the total number of $p-$gons with the given points as vertices is $(p-1)!/2$. Among these $p-$gons, exactly one is not self-intersecting, namely $A_0 A_1 A_2 \cdots A_{p-1} A_0$. The others have at least one pair of sides which cross at an interior point. Therefore the number of self-intersecting polygons is $(p-1)!/2 - 1$. This is not the answer to the problem because some of these polygons can be obtained from others by rotating the circle.
We will say that two polygons P and Q are equivalent if they can be obtained from each other by rotating the circle; we then write $P \sim Q$.
The set of all self-intersecting polygons is now broken up into classes by putting P and Q in the same class whenever $P \sim Q$. The class in which a polygon P lies consists of all polygons equivalent to P; it is called the *equivalence class* of P. Our problem is to determine the number of equivalence classes.
Let P_0 be any polygon, and denote by $P_1, P_2, \cdots, P_{p-1}$ the polygons obtained by rotating P_0 counterclockwise through angles of $360°/p, 2(360°/p), \cdots, (p-1)(360°/p)$. We will prove that $P_0, P_1, \cdots, P_{p-1}$ are either all different or all the same.

Suppose they are not all different, so that $P_i = P_j$, where $0 \le i < j \le p-1$. Putting $k = j - i$, it follows that P_i is unchanged when the circle is rotate through $k \cdot (360°/p)$, where $0 < k < p$. This means that if A_0 is joined to some point A_t is in the polygon P_i, then A_k must be be joined to A_{k+t}, A_{2k} must be joined to A_{2k+t}, etc. We can show that all points $A_0, A_k, A_{2k}, \cdots, A_{(p-1)k}$ are all different. For if $A_{lk} = A_{mk}$, where $0 \le l < m \le p-1$, then $mk - lk = (m-l)k$ must be a multiple of p. This is a contradiction as $0 < k < p, 0 < m - l < p$ and p is a prime.
Thus, every vertex A_s is joined to A_{s+t}, so that the polygon P_i is regular. In this case, it is clear that

$$P_0 = P_1 = \cdots = P_{p-1}.$$

We have now shown that the equivalence class of a non-regular polygon has p members, while that of a regular polygon has only one member. The next step is to determine how many regular self-intersection $p-$gons there are. WE saw that in a regular polygon, each vertex A_s is joined to A_{s+t}, where t is a fixed number satisfying $t \le 1 \le p-1$. But each regular polygon will appear twice in this process, since t and $p-t$ give rise to the same polygon ($t \ne p-t$ as p is odd). Therefore there are $(p-1)/2$ regular polygons; since exactly one of these is non-self-intersecting, there are $(p-1)/2 - 1 = (p-3)/2$ self-intersecting regular polygons.

Now, let N denote the total number of equivalence classes of self-intersecting polygons. Recalling that total number of self-intersecting polygons is $(p-1)!/2 - 1$, we get:

$$\left(\frac{p-3}{2}\right) \cdot 1 + \left(N - \frac{p-3}{2}\right) \cdot p = \frac{(p-1)!}{2} - 1$$

Solving for N,

$$\boxed{N = \frac{1}{2}\left\{\frac{(p-1)!+1}{p} + p - 4\right\}}$$

Q 3.3. The set of positive integers is represented as a union of pairwise disjoint subsets, whose elements form infinite arithmetic progressions with positive differences $d_1,\ d_2,\ d_3,\ \cdots$. Is it possible that the sum

$$\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}+\cdots$$

does not exceed 0.9? Consider the cases where

(a) the total number of progressions is finite, and

(b) the number of progressions is infinite. (In this case, the condition that

$$\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}+\cdots$$

does not exceed 0.9 should be taken to mean that the sum of any finite number of terms does not exceed 0.9.)

Answer: (a) No. (b) Yes.
Let the arithmetic progressions be $D_1,\ D_2,\cdots$, having common differences $d_1,\ d_2,\ \cdots$ respectively. Also, let $a_1,\ a_2,\ \cdots$ be the smallest members of $D_1,\ D_2,\cdots$ respectively.

(a) Assume that there are m arithmetic progressions.
Let N be an integer greater than $a_1,\ a_2,\ \cdots,\ a_m$, and let k_i $(1\le i\le m)$ be the number of members of D_i less than or equal to N. Then

$$a_i+(k_i-1)d_1\le N<a_i+k_id_i,\quad 1\le i\le m.$$

Therefore

$$k_i-1\le\frac{N-a_i}{d_i}<k_i,\quad 1\le i\le m,$$

and so

$$k_i-1\le\frac{N}{d_i}<k_i+\frac{a_i}{d_i},\quad 1\le i\le m.$$

Add all these inequalities. Since

$$N=\sum_{i=1}^{m}k_i,$$

we have

$$N-m\le N\sum_{i=1}^{m}\frac{1}{d_i}<N+\sum_{i=1}^{m}\frac{a_i}{d_i},$$

whence

$$1-\frac{m}{N}\le\sum_{i=1}^{m}\frac{1}{d_i}<1+\frac{1}{N}\sum_{i=1}^{m}\frac{a_i}{d_i}.$$

Finally, let $N\longrightarrow\infty$. Then

$$1\le\sum_{i=1}^{m}\frac{1}{d_i}\le 1,$$

since m and $\sum_{i=1}^{m}\frac{a_i}{d_i}$ are constants. Therefore

$$\sum_{i=1}^{m}\frac{1}{d_i}=1.$$

No, it cannot be less than 0.9.

(b) We will not only show that the result from (a) does not analogously follow but in fact, given $\epsilon > 0$, there exists a selection $D_1,\ D_2,\ \cdots$ such that

$$\sum_{i=1}^{\infty} \frac{1}{d_i} < \epsilon.$$

We produce an example. Let n be a positive integer greater than 1. Make the following inductive hypothesis:
$P(k)$: There exist disjoint arithmetic progressions

$$D_1,\ D_2,\ \cdots,\ D_k$$

with common differences $n,\ n^2,\ \cdots,\ n^k$ whose union contains $\{1,\ 2,\ \cdots, k\}$.

We first note that $P(1)$ is true, letting

$$D_1 = \{x \in \mathbb{Z}^+ \mid x \equiv 1 \mod n\}.$$

Now, suppose that $P(k)$ is true for $k \geq 1$. Now, $D_1 \cup D_2 \cup \cdots \cup D_k$ does not exhaust all the integers. If it did, we would have

$$\sum_{i=1}^{k} \frac{1}{d_i} = \sum_{i=1}^{k} \frac{1}{n^i} = \frac{1}{n} \cdot \frac{1 - \frac{1}{n^k}}{1 - \frac{1}{n}} < \frac{1}{n-1} \leq 1,$$

contradicting (a).
Let a_{k+1} be the smallest positive integer not in $D_1 \cup D_2 \cup \cdots \cup D_k$. Define

$$D_{k+1} = \left\{x \in \mathbb{Z}^+ \mid x \equiv a_{k+1} \mod n^{k+1}, x \geq a_{k+1}\right\}$$

Then $D_i \cap D_{k+1} = \emptyset$ for $1 \leq i \leq k$ since, if not, a_{k+1} is congruent modulo n^i to an element of D_i. But $a_{k+1} > a_i$, the least element of D_i, because a_i is, by induction, the least positive integer not in $D_1 \cup D_2 \cup \cdots \cup D_{i-1}$. Moreover, $a_{k+1} \geq k+1$ since

$$\{1,\ 2,\ \cdots,\ k\} \subset D_1 \cup D_2 \cup \cdots \cup D_k$$

by induction. If $a_{k+1} = k+1$, then $a_{k+1} \in D_{k+1}$. Else, $k+1 \in D_1 \cup D_2 \cup \cdots \cup D_k$. In any case, $k+1 \in D_1 \cup D_2 \cup \cdots \cup D_{k+1}$.
This completes the induction, so $P(k)$ is true for all $k \geq 1$.
We immediately have that

$$\mathbb{Z}^+ = \bigcup_{i}^{\infty} D_i$$

since

$$\{1,\ 2,\ \cdots,\ k\} \subset \bigcup_{i}^{k} D_i$$

for all k.
Finally, observe that

$$\sum_{i=1}^{\infty} \frac{1}{d_i} = \frac{1}{n-1}.$$

Given $\epsilon > 0$, choose $n > \left\lfloor \frac{1}{\epsilon} + 1 \right\rfloor$ and we are done.

From this, it immediately follows that it is indeed possible for the sum to not exceed 0.9.

Q 3.4. A table has m rows and n columns where m and n are positive integers greater than 1. The following permutations of its mn elements are permitted: an arbitrary permutation leaving each element in the same row (a "horizontal move") and an arbitrary permutation leaving each element in the same column (a "vertical move"). Find the number k such that any permutation of mn can be obtained by l permitted moves but there exists a permutation that cannot be achieved in less than k moves.

Answer: The required value of k is three.
To see that $k > 2$, consider the permutation

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad \longrightarrow \quad \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array}$$

As $m, n \geq 2$, we can find a 2×2 subtable in any such $m \times n$ table.
Since b and c have to swap both their row and column positions, at least two "moves" are needed, one "within rows" and one "within columns". But a row move must move at least one of a or d and a column move can not restore it (or them) to its (their) original positions. The same would be true if the column move were done first. So two moves are insufficient. [That three moves are enough will follow from the general argument below.]

One way to achieve any permutation in three moves can be derived as follows: First label every cell in the table with the row number of the cell to which it is to be moved under the given permutation, then carry out the following:

Move 1 (Within rows) Permute each row so that each column contains cells labelled with a complete set of row numbers. [That this can always be achieved is the main burden of proof, which we shall do below.]

Move 2 (Within columns) Permute each column so that the labels on all the cells in the first row are 1, all in the second are 2, and so on. [This can obviously be done, given Move 1.]

Move 3 (Within rows) Permute each row so that every cell is where the given permutation requires. [Given that Moves 1 and 2 have shifted every cell to its correct destination row, Move 3 can obviously be done.]

It must now be established that Move 1 is possible. The labelling fills the mn cells of the table with a random pattern of n 1s, n 2s, ... and n ms. It will presently be established that despite the randomness of this pattern, it is always possible to choose one cell from each row so that the resulting collection of cells is labelled $1, 2, \ldots, m$ in some order. $(*)$
Let Π_n be the row move which swaps each of these cells with the last cell in its row. After Π_n is applied to the table, column n has the required property and columns $1, \ldots, n-1$ form an $m \times (n-1)$ table filled with a random pattern of $(n-1)$ 1s, $(n-1)$ 2s, ..., and $(n-1)$ ms.
By the same argument there is then a row move Π_{n-1} which gives column $n-1$ of this reduced table the required property and leaves columns 1 to $n-2$ forming a new table to which the same process can be applied, and so on. The composite permutation $\Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_n$ (first do Π_n, then Π_{n-1} et cetera) can be considered as acting on the whole original $m \times n$ table and is still a row move; so it is the required Move 1.

$(*)$ It remains to prove the crucial assertion "from an $m \times n$ table filled with n copies of i for $i = 1, \ldots, m$ it is possible to select a different number from each row". Here is a recursive algorithm for the selection process: Select any number from row 1. Having selected distinct numbers from row 1 to $k-1$ make the selection for rows 1 to k as follows: If row k contains a number not already selected from earlier rows, then extend the selection for rows 1 to $k-1$ by selecting that number for row k. If, however, all numbers in row k have already been selected, (say from rows $x_1, x_2, \ldots, x_s$), then somewhere in these rows there must be a number, say N, which has not yet been selected, because rows x_1 to x_s and row k must contain at least $s+1$ different numbers (recall there are only n copies of each number in the whole table and there are n cells in each row). Suppose that row x_t ($t \leq s$) contains this number N, and that the number currently selected from row x_t is M (which belongs to row k as well). Then the required selection for rows 1 to k is: M from row k, N from row x_t and selections from the remaining rows as before. This completes the proof.

4 Challenge

Q 4.1. (Number Theory)
Let S be a subset of positive integers.
n belongs to S if and only if there exists a circle in the XY plane which has exactly n lattice points in its interior (excluding the boundary).
Describe the set S.

Answer: The set S is in fact, $\mathbb{N}$.
We shall show that given any $n \in \mathbb{N}$, there exists a circle with exactly n lattice points in its interior.
In fact, we construct a family of concentric circles with this property.

First, we shall prove the following claim:
Given two distinct lattice points, their distances from $P = \left(\frac{22}{7}, \pi\right)$ are also distinct.
Proof. Let (x_1, y_1) and (x_2, y_2) be two lattice points such that they are are equal distances from P.
We shall show that this is the case if and only if $x_1 = x_2$ and $y_1 = y_2$.

$$\left(x_1 - \frac{22}{7}\right)^2 + (y_1 - \pi)^2 = \left(x_2 - \frac{22}{7}\right)^2 + (y_2 - \pi)^2$$

Case 1. $y_1 = y_2$.
Thus, we have:

$$\left(x_1 - \frac{22}{7}\right)^2 = \left(x_2 - \frac{22}{7}\right)^2$$
$$\implies x_1 = x_2 \text{ or } x_1 + x_2 = \frac{44}{7}$$

As the latter is not possible if x_1 and x_2 are integers, we have it that $y_1 = y_2 \implies x_1 = x_2$.

Case 2. $y_1 \neq y_2$.

$$\left(x_1 - \frac{22}{7}\right)^2 + (y_1 - \pi)^2 = \left(x_2 - \frac{22}{7}\right)^2 + (y_2 - \pi)^2$$

$$\implies x_1^2 - \frac{44}{7}x_1 + y_1^2 + 2\pi y_1 = x_2^2 - \frac{44}{7}x_2 + y_2^2 + 2\pi y_2$$

$$\implies \pi = \frac{x_1^2 - x_2^2 - \frac{44}{7}(x_1 - x_2) + y_1^2 - y_2^2}{2(y_2 - y_1)}$$

This is a contradiction as π is not rational.
Thus, we have it that $x_1 = x_2$ and $y_1 = y_2$. ■
Now it is easy to see that increasing the radius of a circle with center at P will cover lattice points one by one.

Q 4.2. (Functional Equation)
Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ and $g(x) = x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$ be two polynomials with real coefficients such that for each real number $x, f(x)$ is the square of an integer if and only if so is $g(x)$.
Prove that if $n + m > 0$, there exists a polynomial $h(x)$ with real coefficients such that $f(x) \cdot g(x) = (h(x))^2$.

As f' is a polynomial with positive leading coefficients, there is a point after which it is positive for every real x. Similarly, there exists a point for f as well. Thus, there is a point a such that f is both increasing and positive for all $x \geq a$.
Similarly, there exists such a point for g.
WLOG, assume that $a \geq b$.
So, after $x = a$, both f and g are increasing and positive.
As f is continuous, $f(x)$ hits all integral squares after $f(a)$ exactly once. At each of these points $g(x)$ is also a square. To make it more precise, there exists a sequence of points such that:
$r < r_i < r_{i+1} < r_{i+2} < \cdots$ so that $f(r_k) = k^2$ for $k \geq i$. Then, $g(r_i) = (i+j)^2$ for some $j \geq -i$.
Then, $g(r_{i+1}) = (i+1+j)^2$, because if it were any other square, then either g won't be increasing or there'd exist a number x between r_i and r_{i+1} so that $g(x) = (i+1+j)^2$ and $f(x)$ is not a perfect square.
Likewise, $f(r_{i+k}) = (i+k)^2$ and $g(r_{i+k}) = (i+k+j)^2$ for all $k \geq 0$.

Consider the polynomial $P(x) := \left(\frac{g(x) + f(x) - j^2}{2}\right)^2$. At $x = r_k$ for $k \geq i$, $P(x)$ equals

$$\left(\frac{g(r_k) + f(r_k) - j^2}{2}\right)^2 = \left(\frac{(k+j)^2 + k^2 - j^2}{2}\right)^2 = \left(\frac{2k^2 + 2kj}{2}\right)^2$$

$$= k^2(k+j)^2 = f(r_k)g(r_k).$$

So, for an infinite number of points, $P(x)$ equals $f(x)g(x)$, and hence it is identically equal to $f(x)g(x)$.
So, we can let $h(x) := \frac{g(x) + f(x) - j^2}{2}$, and we are done.

Q 4.3. (Combinatorics)
Amy and Sheldon play a game in the following manner:
Amy picks a positive integer N and tells Sheldon.
Sheldon then chooses a collection of eleven (not necessarily distinct) integers $(a_i)_{i=1}^{11}$.
Now, Amy must choose a collection $(b_i)_{i=1}^{11}$ where each b_i is either $-1,\ 0,\ 1$ and at least one b_i is non-zero.
After this, the sum $S = \sum_{i=1}^{11} a_i b_i$ is computed. Amy wins the game iff S is divisible by N, otherwise Sheldon wins.
Assuming Amy and Sheldon play optimally, what is the largest value of N that Amy can pick such that she can win?

Answer: 2047 or $2^{11} - 1$

Claim 1: Amy cannot win if $N \geq 2^{11}$.

Proof. As Sheldon plays optimally, he can choose the collection $a_i = 2^{(i-1)}$ for $1 \leq i \leq 11$.
Now, we will show that no valid collection (b_i) can lead to Amy's win. This can be demonstrated as follows:
The maximum value of S is obtained if $b_i = 1$ for all i. Thus, $S_{\max} = 1 + 2 + \cdots + 2^{10} = 2^{11} - 1$.
Similarly, $S_{\min} = -\left(2^{11} - 1\right)$.
This means that for any collection (b_i), $|S| < 2^{11} \leq N$.
Thus, S is divisible by N iff $S = 0$.

$$\implies b_1 + 2b_2 + 2^2 b_3 + \cdots + 2^{10} b_{11} = 0.$$

By looking at the remainders mod 2 on both sides, we get that $b_1 = 0$. This gives us that:

$$\implies 2b_2 + 2^2 b_3 + \cdots + 2^{10} b_{11} = 0.$$

Similarly, by considering the remainders mod 2^2 give us that $b_2 = 0$.
Proceeding in this manner will give us that $b_1 = b_2 = \cdots = b_{11} = 0$. A contradiction. □

By the above claim, we have it that $N \leq 2^{11} - 1$. We will now show that $N = 2^{11} - 1$ actually guarantees a win for Amy.

Claim 2: Amy wins if $N = 2^{11} - 1$.

Proof. Let $(a_i)_{i=1}^{11}$ be any arbitrary collection of integers that Sheldon could have chosen.
Consider all possible collections $(c_i)_{i=1}^{11}$ where each c_i is either 0 or 1.
There are 2^{11} such collections. Compute the sum $S' = \sum_{i=1}^{11} a_i c_i$ for all such collections.
There are 2^{11} (not necessarily distinct) such values of S'.
Consider the values $S' \mod N$. There are only $N = 2^{11} - 1$ such values. Thus, by the pigeonhole principle, there are two distinct collections (c_i') and (c_i'') such that the sums S' corresponding to them give the same remainder.
Formally, we have that $\sum_{i=1}^{11} a_i(c_i' - c_i'') \equiv 0 \mod N$.

Let $b_i = c_i' - c_i''$ for each $1 \leq i \leq 11$. Thus, $\sum_{i=1}^{11} a_i b_i \equiv 0 \mod N$, as desired. This is a valid choice of collection (b_i) as $b_i = -1, 0, 1$ for each valid i. Moreover, as c_i' and c_i'' are distinct, there is at least one i such that $b_i \neq 0$, completing our proof. □

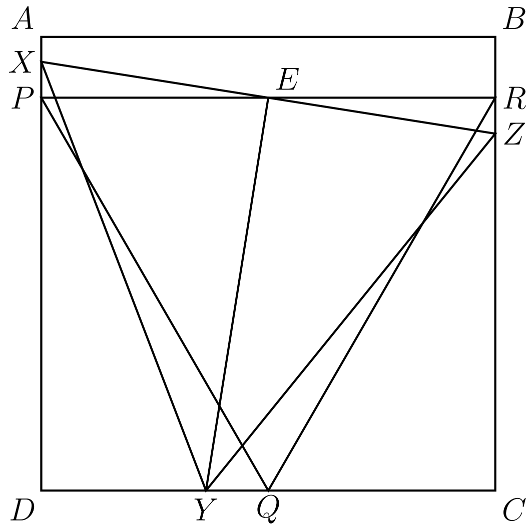
By the above two claims, we have it that $N = 2^{11} - 1 = \boxed{2047}$ is the largest such N.

Q 4.4. (Geometry)

On a plane, a square is given, and 2019 equilateral triangles are inscribed in this square. All vertices of any of these triangles lie on the border of the square. Prove that one can find a point on the plane belonging to the borders of no less than 505 of these triangles.

Let $ABCD$ be the square. The equilateral triangles inscribed in it can be classified as (AB, BC, CD), (BC, CD, DA), (CD, DA, AB) or (DA, AB, BC) according to which three sides of the square contain their vertices.

A triangle with a vertex at A will come under both (AB, BC, CD) and (BC, CD, DA) and similarly for triangles with a vertex at B, C or D. Since there are 2019 triangles and four classes, one of the classes, say (BC, CD, DA), contains at least 505 triangles. Let Q be the midpoint of CD. Let P on AD and R on BC be such that PQR is an equilateral triangle. Let E be the midpoint of PR.



We claim that if XYZ is an equilateral triangle with X on AD, Y on CD and Z on BC, then E is the midpoint of XZ. Join Y and draw the line through E perpendicular to EY, cutting AD at X' and BC at Y'.

Note that EPX' and EQY are similar triangles. Hence $EX'/EY = EP/EQ$, so that $EX'Y'$ and EPQ are also similar triangles. It follows that $EX'Y$ is half an equilateral triangles, and EYZ' is the other half by a similar argument. Since the equilateral triangle XYZ is uniquely determined by Y, we must have $X' = X$ and $Z' = Z$.

This justifies our claim as E must be the midpoint of XZ. It follows that E lies on the perimeter of every triangle in the class (BC, CD, DA), which contains at least 505 triangles.

Q 4.5. (Inequality)
The numbers $1, 2, \ldots, N$ are written on a board where N is a positive integer strictly greater than 1729. Sheldon performs an operation where he erases four numbers of the form $a,\ b,\ c,\ a+b+c$ and then writes $a+b,\ b+c,\ c+a$ in their place.
Prove that he can do this operation no more than $\left\lfloor \dfrac{N(N-1)}{2(2N+1)} \right\rfloor$ times.

Note the two identities:
$a+b+c+(a+b+c) = (a+b)+(b+c)+(c+a)$ and $a^2+b^2+c^2+(a+b+c)^2 = (a+b)^2+(b+c)^2+(c+a)^2$.
From these, we get that the sum of numbers on the board and the sum of the squares of the numbers on the board are invariants.
More specifically, the sum of numbers on the board is always $\dfrac{N(N+1)}{2}$ and the sum of squares is always $\dfrac{N(N+1)(2N+1)}{6}$.

Suppose that there are n numbers $(a_1, a_2, \ldots, a_n)$ on the board at a given instant, we know that:

$$\left(\frac{a_1 + a_2 + \cdots a_n}{n}\right)^2 \leq \frac{a_1^2 + a_2^2 + \cdots a_n^2}{n}$$

Thus, we have

$$\left(\frac{N(N+1)}{2n}\right)^2 \leq \frac{N(N+1)(2N+1)}{6n}.$$

Solving for n gives us

$$\left(\frac{3}{2} \cdot \frac{N(N+1)}{2N+1}\right) \leq n.$$

This gives us that

$$N - n \leq N - \left(\frac{3}{2} \cdot \frac{N(N+1)}{2N+1}\right) = \frac{N(N-1)}{2(2N+1)}.$$

As each operation reduces the number of numbers by 1, we have it that Sheldon can do at most $\left\lfloor \dfrac{N(N-1)}{2(2N+1)} \right\rfloor$ operations.

Q 4.6. (Probability) Let $n \geq 4$ be given, and suppose that the points $P_1, P_2, \ldots, P_n$ are randomly chosen on a circle. Consider the convex $n-$gon whose vertices are these points. What is the probability that at least one of the vertex angles of this polygon is acute?

The angle at the vertex P_i is acute if and only if all other points lie on an open semicircle facing P_i. We first deduce from this that if there are any two acute angles at all, they must occur consecutively. Otherwise, the arcs that these angles subtend would overlap and cover the whole circle, and the sum of the measures of the two angles would exceed 180°.
So the polygon either has just one acute angle or two consecutive acute angles. In particular, taken in counterclockwise order, there exists *exactly* one pair of consecutive angles the second which is acute and the first of which is not.
We are left with the computation of the probability that for one of the points P_j, the angle at P_j is not acute, but the following angle is. This can be done using integrals. But there is a clever argument that reduces the geometric probability to a probability with a finite number of outcomes. The idea is to choose randomly $n-1$ pairs of antipodal points, and then among these to choose the vertices of the polygon. A polygon with one vertex at P_j and the other among these points has the desired property exactly when $n-2$ vertices lie on the semicircle on the clockwise side of P_j and one vertex on the opposite semicircle. Moreover, the points on the semicircle should include the counterclockwise-most to guarantee that the angle at P_j is not acute. Hence there are $n-2$ favourable choices of the total 2^{n-1} choices of points from the antipodal pairs. The probability for obtaining a polygon with the desired property is therefore $(n-2)2^{-n+1}$.
Integrating over all choices of pairs of antipodal pairs preserves the ratio. The events $j = 1, 2, \ldots, n$ are independent, so the probability has to be multiplied by n. The answer to problem is therefore $n(n-2)2^{-n+1}$.